Concordia University
John Molson
School of Business

COMM 225: Production and Operations Management

Fall 2010, Section B

TOPIC: SOLUTIONS TO LINEAR PROGRAMMING – QUESTIONS

(Solved in Class)

PS: Note that the notations might be different from what was used in the class.

Q1 (Resource Allocation): *The program manager for Channel 10 would like to determine the best way to allocate the time for the 11:00 - 11:30 evening news broadcast. Specifically, she would like to determine the number of minutes of broadcast time to devote to local news, national news, weather, and sports. Over the 30-minute time-slot, 10 minutes are set aside for advertising and do not count as broadcast time. The station's policy states that:*

- *at least 15% of the broadcast time should be devoted to local news coverage;*
- *the time devoted to local or national news (combined) must be at least 50% of the broadcast time;*
- *the time devoted to the weather segment must be less than equal to the time devoted to the sports segment;*
- *the time devoted to the sports segment should be no longer than the total time spent on local and national news (combined); and*
- *at least 20% of the broadcast time should be devoted to the weather segment.*

The production costs/minute are $300 for local, $200 for national, $100 for weather, and $100 for sports. Formulate a model that would help the production manager determine the broadcast time to devote to various programs.

Solution: Let X_L, X_N, X_W and X_S denote the time devoted to local news, national news, weather, and sports respectively. Note that over the 30-minute time-slot, 10 minutes are set aside for advertising and do not count as broadcast time. Hence, the total broadcast time available is 20 minutes.

Minimize Total Cost = $Z = 300X_L + 200X_N + 100X_W + 100X_S$

Subject to:

(1) At least 15% of the broadcast time should be devoted to local news coverage;

$X_L \geq 0.15 * 20$

(2) The time devoted to local or national news (combined) must be at least 50% of the broadcast time;

$X_L + X_N \geq 0.5 * 20$

(3) The time devoted to the weather segment must be less than or equal to the time devoted to the sports segment;

$X_W \leq X_S$

(4) The time devoted to the sports segment should be no longer than the total time spent on local and national news (combined); and

$X_S \leq X_L + X_N$

(5) At least 20% of the broadcast time should be devoted to the weather segment.

$X_W \geq 0.20 * 20$

(6) The total broadcast time available is 20 minutes.

$X_L + X_N + X_W + X_S = 20$

(7) All variables are non-negative.

$X_L, X_N, X_W, X_S \geq 0$

Q2 (Blending): *The mystic coffee shop blends coffee on the premises for its customers. It sells three basic blends in one-pound bags: Special, Mountain Dark, and Mill Regular. It uses four different types of coffee to produce the three blends: Brazilian, Mocha, Columbian, and Mild. The shop used the following blend recipe requirements:*

Blend	*Mix Requirements*	*Selling Price/lb*
Special	*At least 40% Columbian; At least 30% Mocha*	*$6.50*
Dark	*At least 60% Brazilian; No more than 10% Mild*	*$5.25*
Regular	*At least 30% Brazilian; No more than 60% Mild*	*$3.75*

The cost of Brazilian coffee is $2 per pound, the cost of Mocha is $2.75 per pound, the cost of Columbian is $2.90 per pound, and the cost of Mild is $1.70 per pound. The shop has 110 pounds of Brazilian coffee, 70 pounds of Mocha, 80 pounds of Columbian and 150 pounds of Mild coffee available per week. The shop wants to know the amount of each blend it should prepare each week in order to maximize profit.

Solution: Let S, D, and R stand for Special, Dark, and Regular blend of coffee. Also let B, M, C, and L denote the Brazilian, Mocha, Columbian, and Mild coffee. Let X_{ij} = amounts (lbs) of coffee "i" used in the production of blend "j". The list of variables is as follows:

	Blend		
Coffee	Special	Dark	Regular
Brazilian	X_{BS}	X_{BD}	X_{BR}
Mocha	X_{MS}	X_{MD}	X_{MR}
Colombian	X_{CS}	X_{CD}	X_{CR}
Mild	X_{LS}	X_{LD}	X_{LR}

Maximize profit

$$Z = 6.5(X_{BS} + X_{MS} + X_{CS} + X_{LS}) + 5.25(X_{BD} + X_{MD} + X_{CD} + X_{LD}) + 3.75(X_{BR} + X_{MR} + X_{CR} + X_{LR}) - 2.0(X_{BS} + X_{BD} + X_{BR}) - 2.75(X_{MS} + X_{MD} + X_{MR}) - 2.90(X_{CS} + X_{CD} + X_{CR}) - 1.70(X_{LS} + X_{LD} + X_{LR})$$

Subject to:

(1) The shop has 110 pounds of Brazilian coffee, 70 pounds of Mocha, 80 pounds of Columbian and 150 pounds of Mild coffee available per week.

$$X_{BS} + X_{BD} + X_{BR} \leq 110$$
$$X_{MS} + X_{MD} + X_{MR} \leq 70$$
$$X_{CS} + X_{CD} + X_{CR} \leq 80$$
$$X_{LS} + X_{LD} + X_{LR} \leq 150$$

(2) Special blend had at least 40% Columbian and at least 30% Mocha

$$X_{CS} \geq 0.40(X_{BS} + X_{MS} + X_{CS} + X_{LS})$$
$$X_{MS} \geq 0.30(X_{BS} + X_{MS} + X_{CS} + X_{LS})$$

(3) Dark blend had at least at least 60% Brazilian and no more than 10% Mild

$$X_{BD} \geq 0.60(X_{BD} + X_{MD} + X_{CD} + X_{LD})$$
$$X_{LD} \leq 0.10(X_{BD} + X_{MD} + X_{CD} + X_{LD})$$

(4) Regular blend had at least at least 30% Brazilian and no more than 60% Mild

$$X_{BR} \geq 0.30(X_{BR} + X_{MR} + X_{CR} + X_{LR})$$
$$X_{LR} \leq 0.60(X_{BR} + X_{MR} + X_{CR} + X_{LR})$$

(5) All variables are non-negative.

Q3 (Production Planning): *Paper can be made from new wood pulp, from recycled office paper, or from recycled newsprint. New pulp costs $100 per ton, recycled office paper $50 per ton, and recycled newsprint, $20 per ton. Four different processes can be used to produce paper. To produce one ton of paper:*

- *Process 1 uses 3 tons of wood pulp*
- *Process 2 uses 1 ton of wood pulp and 4 tons of recycled office paper*
- *Process 3 uses 1 ton of wood pulp and 12 tons of recycled newsprint*
- *Process 4 uses 8 tons of recycled office paper.*

At the moment only 80 tons of pulp is available. The company would like to produce 100 tons of new paper at minimum total cost. Formulate a model that would help the production manager determine the production plan.

Solution: Let X_i *denote the amount (in tons) of new paper produced using process "I".*

Let Y_{NP}, Y_{RO}, Y_{RN} *denote the amount (in tons) of new wood pulp, recycled paper, and recycled newsprint used.*

Minimize Total Cost = $Z = 100Y_{NP} + 50Y_{RO} + 20Y_{RN}$

Subject to:

(1) The total amount of paper produced should be 100 tons;

$X_1 + X_2 + X_3 + X_4 = 100$

(2) The total amount of new wood pulp required is

$3X_1 + X_2 + X_3 = Y_{NP}$

(3) The total amount of recycled office paper required is

$4X_2 + 8X_4 = Y_{RO}$

(4) The total amount of recycled newsprint required is

$12X_3 = Y_{RN}$

(5) At the moment only 80 tons of new wood pulp is available.

$Y_{NP} \le 80$

(6) All variables are non-negative.

Q4 (Blending): *Daisy Drugs manufactures two drugs: 1 and 2. the drugs are produced by blending two chemicals – 1 and 2.*

- *By weight, drug 1 must contain at least 65% chemical 1 and drug 2 must contain at least 55% of chemical 1.*
- *Drug 1 sells for $6/oz, and drug 2 sells for $4/oz. Chemical 1 and 2 can be produced by one of two production processes.*
- *Running process 1 for an hour requires 3 oz of raw material and 2 hours of skilled labor, and yields 3 oz of chemical 1 and 3 oz of chemical 2.*
- *Running process 2 for an hour requires 2 oz of raw material and 3 hours of skilled labor, and yields 3 oz of chemical 1 and 1 oz of chemical 2.*
- *A total of 120 hours of skilled labor and 100 oz of raw materials are available.*

Formulate an LP that can be used to maximize Daisy's Sales revenues.

Solution:

Let X_{11} denote the amount of chemical 1 that goes into drug 1, X_{12} denote the amount of chemical 1 that goes into drug 2, X_{21} denote the amount of chemical 2 that goes into drug 1, and X_{22} denote the amount of chemical 2 that goes into drug 2. All the quantities are in ounces.

Maximum Total profit $Z = 6(X_{11} + X_{21}) + 4(X_{12} + X_{22})$

Constraints:

(1) Mix requirements constraints:

- Drug 1 must contain at least 65% chemical 1:

  $X_{11} \geq 0.65(X_{11} + X_{21})$ Or $X_{11} - 0.65(X_{11} + X_{21}) \geq 0$ (1)

- Drug 2 must contain at least 55% of chemical 1:

  $X_{12} \geq 0.55(X_{12} + X_{22})$ Or $X_{12} - 0.55(X_{12} + X_{22}) \geq 0$ (2)

(2) Resource constraints:

$3P_1 + 2P_2 \leq 100$ (Raw Material) (3)

$2P_1 + 3P_2 \leq 120$ (Skilled Labor) (4)

(3) Amounts of chemical used should be less than the amounts of chemical produced.

$X_{11} + X_{12} \leq 3P_1 + 3P_2$ Or $X_{11} + X_{12} - 3P_1 - 3P_2 \leq 0$ (3)

$X_{21} + X_{22} \leq 3P_1 + P_2$ Or $X_{21} + X_{22} - 3P_1 - P_2 \leq 0$ (4)

(4) All variables are non-negative.

Q5 (Product Mix): *Red Dwarf Toasters needs to produce 1000 of their new "Talking toasters". There are three ways this toaster can be produced: manually, semi-automatically, and robotically.*

(1) Manual assembly requires 1 minute of skilled labor, 40 minutes of unskilled labor, and 3 minutes of assembly room time. The corresponding values for semiautomatic assembly are 4, 30 and 2; while those for robotic assembly are 8, 20 and 4.

(2) There are 4500 minutes of skilled labor, 36,000 minutes of unskilled labor, and 2700 minutes of assembly room time available for this product.

(3) The total cost for producing manually is $ 7/toaster; semi-automatically is $8/toaster; and $8.50/toaster.

(4) The union contract states that the amount of skilled labor time used is at least 10% of the total labour (skilled and unskilled) time used.

Formulate the problem of producing 1000 toasters at minimum cost meeting the resource requirements. Clearly define your decision variables, parameters, objectives, and constraints.

Solution: Let X_M, X_S, and X_R denote the number of toasters produced manually, semi-automatically, and robotically respectively. The resource information is as follows:

Methods	***Skilled Labor used (min)***	***Unskilled labor used (min)***	***Assembly room time used (min)***
Manually	*1*	*40*	*3*
Automatically	*4*	*30*	*2*
Semi-automatically	*8*	*20*	*4*
Total available	*4500 min*	*36,000 min*	*2700 min*

Minimize Total Cost = $Z = 7X_M + 8X_S + 8.5X_R$

Subject to:

(1) The total number of toasters produced should be 1000 units;

$X_M + X_S + X_R = 1000$

(2) The total amount of skilled labor used should be less than what is available;

$X_M + 4X_S + 8X_R \leq 4500$

(3) The total amount of unskilled labor used should be less than what is available;

$40X_M + 30X_S + 20X_R \leq 36000$

(4) The total amount of assembly time used should be less than what is available;

$3X_M + 2X_S + 4X_R \leq 2700$

(5) The union contract states that the amount of skilled labour time used is at least 10% of the total labour (skilled and unskilled) time used.

$X_M + 4X_S + 8X_R \geq 0.10(X_M + 4X_S + 8X_R + 40X_M + 30X_S + 20X_R)$

(6) All variables are non-negative.

$X_M, X_S, X_R \geq 0$

Q6 (Multi-period Production Planning with Inventory Costs): *IT Computer Services assembles its own brand of personal computers from component parts it purchases overseas and domestically. IT sells most of its computers locally to different departments at State University as well as to individuals and businesses in the intermediate geographic region. IT has enough regular production capacity to produce 160 computers per week. IT can produce additional 50 computers with overtime. The cost of assembly, inspection, and packaging a computer during regular time is $190. Overtime production of a computer costs $260. Further it costs $5 per computer per week to hold a computer in inventory for future delivery. IT wants to be able to meet all customer orders with no shortages in order to provide quality service. The order schedule for the next 6 weeks is as follows:*

Week	*1*	*2*	*3*	*4*	*5*	*6*
Computer Orders	*105*	*170*	*230*	*180*	*150*	*250*

IT Computers wants to determine a schedule that will indicate how much regular and overtime production it will need each week in order to meet its orders at the minimum cost. The company wants no inventory left over at the end of the six-week period. Formulate a linear programming model for this problem.

Solution: Let X_i be the number of computers produced with regular hours in week "i"; Y_i: be the number of computers produced with overtime in week "i"; and I_i: be the number of computers in inventory at the end of week "i".

Week	*Production during regular time*	*Production during overtime*	*Inventory at the end of the week*	*Demand*
1	X_1	Y_1	I_1	*105*
2	X_2	Y_2	I_2	*170*
3	X_3	Y_3	I_3	*230*
4	X_4	Y_4	I_4	*180*
5	X_5	Y_5	I_5	*150*
6	X_6	Y_6	I_6	*250*

Minimize Total Costs =

$$Z = 190(X_1 + X_2 + X_3 + X_4 + X_5 + X_6) + 260(Y_1 + Y_2 + Y_3 + Y_4 + Y_5 + Y_6) + 5(I_1 + I_2 + I_3 + I_4 + I_5 + I_6)$$

Subject to:

(1) During the next six weeks, the demand should be met;

$I_1 = I_0 + X_1 + Y_1 - 105$

$$I_2 = I_1 + X_2 + Y_2 - 170$$
$$I_3 = I_2 + X_3 + Y_3 - 230$$
$$I_4 = I_3 + X_4 + Y_4 - 180$$
$$I_5 = I_4 + X_5 + Y_5 - 150$$
$$I_6 = I_5 + X_6 + Y_6 - 250$$

(2) IT has enough regular production capacity to produce 160 computers per week. IT can produce additional 50 computers with overtime.

$X_1 \le 160;$ $\quad$ $Y_1 \le 50;$

$X_2 \le 160;$ $\quad$ $Y_2 \le 50;$

$X_3 \le 160;$ $\quad$ $Y_3 \le 50;$

$X_4 \le 160;$ $\quad$ $Y_4 \le 50;$

$X_5 \le 160;$ $\quad$ $Y_5 \le 50;$

$X_6 \le 160$ $\quad$ $Y_6 \le 50$

(3) At the beginning of week 1, there is no inventory in stock. The company wants no inventory left over at the end of the six-week period.

$$I_0 = 0$$
$$I_6 = 0$$

(4) All variables are non-negative.

Q7 (Multi-period Production Planning with Inventory Costs): *During the next three months, Airco must meet the following demands for air conditioners: month 1, 300; month 2, 400; month 3, 500. Air conditioners can be produced in either New York or Los Angeles. It takes 1.5 hours of skilled labor to produce an air conditioner in Los Angeles, and 2 hours in New York. It costs $400 to produce an air conditioner in Los Angeles, and $350 in New York. During each month, each city has 420 hours of skilled available. It costs $100 to hold an air conditioner in inventory at a centralized location for a month. At the beginning of month 1, Airco has 200 conditioners in stock. Formulate a mathematical model whose solution would help Airco minimize the cost of meeting air-conditioner demands for the next three months.*

Solution:

Let X_{LA1} , X_{LA2}, X_{LA3}, X_{NY1}, X_{NY2}, *and* X_{NY3} *denote the amount number of air-conditioners produced in month 1, 2, and 3 at LA and NY respectively. Let* I_1 *be the number of units of air-conditioners in inventory at the end of month 1.*

Minimize Total Costs =

$$Z = 400(X_{LA1} + X_{LA2} + X_{LA3}) + 350(X_{NY1} + X_{NY2} + X_{NY3}) + 100(I_1 + I_2 + I_3)$$

Subject to:

(5) During the next three months, Airco must meet the following demands for air conditioners: month 1, 300; month 2, 400; month 3, 500.

$$I_1 = I_0 + X_{LA1} + X_{NY1} - 300$$
$$I_2 = I_1 + X_{LA2} + X_{NY2} - 400$$
$$I_3 = I_2 + X_{LA3} + X_{NY3} - 500$$

(6) At the beginning of month 1, Airco has 200 conditioners in stock. (Note that the inventory at the end of month 3 will be zero as there is a holding cost associated with it).

$I_0 = 200$

(7) It takes 1.5 hours of skilled labour to produce an air conditioner in Los Angeles, and 2 hours in New York. During each month, each city has 420 hours of skilled available.

$1.5X_{LA1} \leq 420$

$1.5X_{LA2} \leq 420$

$1.5X_{LA3} \leq 420$

$2X_{NY1} \leq 420$

$2X_{NY2} \leq 420$

$2X_{NY3} \leq 420$

(8) All variables are non-negative.

Q8 (Investment Problem): *Consider a mortgage team with $100,000,000 to finance various investments. There are five categories of loans, each with an associated return and risk (1-10, 1 best):*

Loan/Investment	***Return (%)***	***Risk***
First Mortgages	*9*	*3*
Second Mortgages	*12*	*6*
Personal Loans	*15*	*8*
Commercial Loans	*8*	*2*
Government Securities	*6*	*1*

Any uninvested money goes into a savings account with no risk and 3% return. The goal for the mortgage team is to allocate the money to the categories so as to:

a) *Maximize the average return per dollar*
b) *Have an average risk of no more than 5 (all averages and fractions taken over the invested money (not over the saving account)).*
c) *Invest at least 20% in commercial loans*
d) *The amount in second mortgages and personal loans combined should be no higher than the amount in first mortgages.*

Solution: Let X_i denote the amount of money invested in option "i".

Maximize Total Return per dollar = $Z = 9X_1 + 12X_2 + 15X_3 + 8X_4 + 6X_5$

Subject to:

(1) The total amount of money available for investment is $100,000,000;

$X_1 + X_2 + X_3 + X_4 + X_5 \leq 100{,}000{,}000$

Note that we cannot use "=" in this constraint as it states that the any uninvested money goes into a savings account with no risk and 3% return.

(2) Have an average risk of no more than 5 (all averages and fractions taken over the invested money (not over the saving account)).

$$\frac{9X_1 + 12X_2 + 15X_3 + 8X_4 + 6X_5}{X_1 + X_2 + X_3 + X_4 + X_5} \leq 5$$

or $9X_1 + 12X_2 + 15X_3 + 8X_4 + 6X_5 \leq 5(X_1 + X_2 + X_3 + X_4 + X_5)$

(3) Invest at least 20% in commercial loans

$X_4 \geq 0.20 * 100,000,000$

(4) The amount in second mortgages and personal loans combined should be no higher than the amount in first mortgages.

$X_2 + X_3 \leq X_1$

(5) All variables are non-negative.

$X_1, X_2, X_3, X_4, X_5 \geq 0$

Q9 (Transportation Problem): *One of the main products of P&T Company is canned peas. The peas are prepared at three canneries (near Bellingham, Washington; Eugene, Oregon; and Albert Lea, Minnesota) and are then shipped by truck to four distributing warehouses in Sacremento, California; Salt Lake City, Utah; Rapid City, South Dakota; and Albuquerque, New Mexico. Because shipping costs are a major expense, management has begun a study to reduce them. For the upcoming season, an estimate has been made of what the output will be from each cannery, and how much each warehouse will require to satisfy its customers. The shipping costs per truckload from each cannery to each warehouse has also been determined. This is summarized in table below. Formulate the problem of meeting the demand at the minimum cost.*

	Warehouse 1	*Warehouse 2*	*Warehouse 3*	*Warehouse 4*	***Supply***
Cannery 1	*$ 464*	*513*	*654*	*867*	*75 truckloads*
Cannery 2	*352*	*416*	*690*	*791*	*125 truckloads*
Cannery 3	*995*	*682*	*388*	*685*	*100 truckloads*
Demand	*80 truckloads*	*65 truckloads*	*70 truckloads*	*85 truckloads*	

Solution: Let X_{ij} denote the number of truckloads shipped from cannery "i" to warehouse "j".

Minimize Total Cost =

$$Z = 464X_{11} + 513X_{12} + 654X_{13} + 867X_{14} + 352X_{21} + 416X_{22} + 690X_{23} + 791X_{24} + 995X_{31} + 682X_{32} + 388X_{33} + 685_{34}$$

Constraints:

Supply Constraints:

(1) $X_{11} + X_{12} + X_{13} + X_{14} \leq 75$

(2) $X_{21} + X_{22} + X_{23} + X_{24} \leq 125$

(3) $X_{31} + X_{32} + X_{33} + X_{34} \leq 100$

Demand Constraints:

(1) $X_{11} + X_{21} + X_{31} \geq 80$

(2) $X_{12} + X_{22} + X_{32} \geq 65$

(3) $X_{13} + X_{23} + X_{33} \geq 70$

(4) $X_{14} + X_{24} + X_{34} \geq 85$

(5) All variables are non-negative.

Q10 (Cutting Stock Problem): *A furniture manufacturer buys wooden strips from a supplier. Each strip is 25 ft. in length. The manufacturer cuts these strips into lengths of 7 ft., 9 ft. and 10 ft. units to use them for its furniture manufacturing. The minimum demand for next week is given as follows:*

7 ft. strips: 500 units; 9 ft strips: 350 units; 10 ft. strips: 600 units

a) Formulate an LP problem for the manufacturer that minimizes the number of strips bought from the supplier.
b) How would the formulation change if the objective is defined as minimization of the trim loss (i.e. part of the strip that is wasted)?

Solution:

Let X_i *denote the number of strips cut as per the configuration "i". The configurations are as follows:*

Configuration	7 ft. strips	9 ft. strips	10 ft. strips	Waste
1	3	0	0	25-21 = 4ft
2	2	1	0	25-14-9 = 2 ft
3	2	0	1	25-14-10 = 1 ft
4	1	2	0	25-14-9 = 0 ft
5	0	1	1	25-19 = 6 ft
6	0	0	2	25-20 = 5 ft

a) ***Minimize Total Number of Strips:*** $Z = X_1 + X_2 + X_3 + X_4 + X_5 + X_6$

Constraints:

(1) Demand for the 7 ft. strip should be met.

$3X_1 + 2X_2 + 2X_3 + X_4 + 0 * X_5 + 0 * X_6 = 500$

(2) Demand for the 9 ft. strip should be met.

$0 * X_1 + 1 * X_2 + 0 * X_3 + 2X_4 + 1 * X_5 + 0 * X_6 = 350$

(3) Demand for the 10 ft. strip should be met.

$0 * X_1 + 0 * X_2 + 1 * X_3 + 0 * X_4 + 1 * X_5 + 2 * X_6 = 600$

(4) All variables are non-negative.

b) ***The objective function will change to:***

Minimize Total Trim Loss: $Z = 4X_1 + 2X_2 + X_3 + 0 * X_4 + 6X_5 + 5X_6$

The constraints remain unchanged.